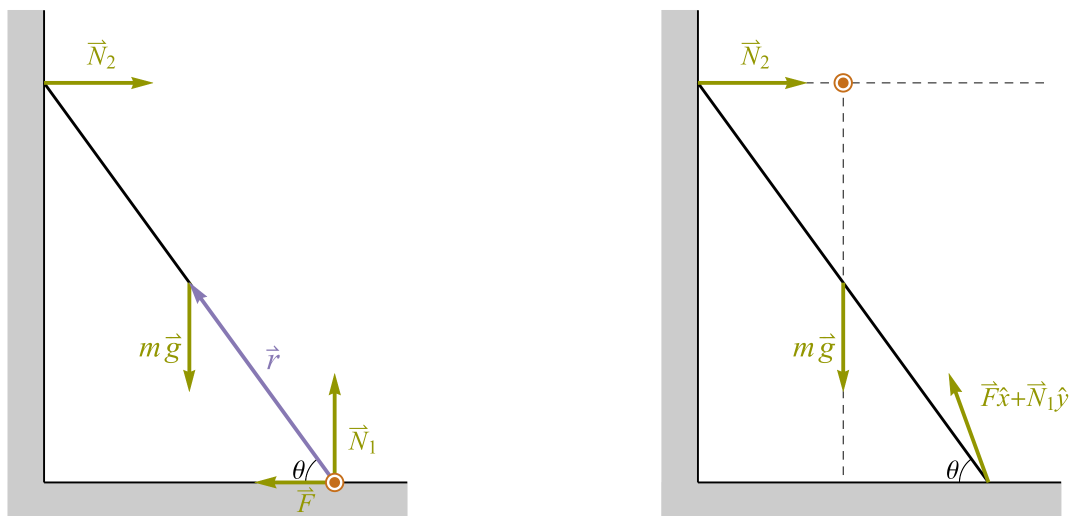
Lecture 3 - Pull!

A Puzzle...

Recall from last time that we computed the stability criterion $\frac{1}{2\mu} \leq \text{Tan}[\theta]$ for a leaning ladder (of length d):

- We computed the stability using the base of the ladder as the torque base point (below, left)
- Redo the problem using the much slicker choice for the torque base point at the intersection of the $\vec{N}_2$ and $m\,\vec{g}$ forces (below, right)



Solution

In total there are three forces acting on the ladder: a normal force at the top end, gravity in the middle, and a third force $-F\,\hat{x} + N_1\,\hat{y}$ at the base of the ladder. Denote the origin (0, 0) as the bottom-left corner. Then extending lines through the normal force at the top of the ladder and gravity's force in the middle yields an intersection point at $\left(\frac{d}{2}\,\text{Cos}[\theta],\, d\,\text{Sin}[\theta]\right)$. What if we put our origin for the calculation of torque here? Because the top normal and gravity pass through this point, their contribution to the overall torque equals 0. And since there is only one other force acting on the ladder $(-F\,\hat{x} + N_1\,\hat{y})$, and the net torque must equal 0, then $-F\,\hat{x} + N_1\,\hat{y}$ must pass through this point as well.

Using similar triangles, this implies that

$$\frac{N_1}{F} = \frac{d\,\text{Sin}[\theta]}{\frac{d}{2}\,\text{Cos}[\theta]} = 2\,\text{Tan}[\theta] \tag{1}$$

so that the ladder will not slide provided

$$F \leq \mu\,N_1 \tag{2}$$

$$\frac{1}{\mu} \leq \frac{N_1}{F} \tag{3}$$

$$\frac{1}{\mu} \leq 2\,\text{Tan}[\theta] \tag{4}$$

$$\frac{1}{2\mu} \leq \text{Tan}[\theta] \quad (5)$$

in agreement with our previous answer! □

Supplementary Section: Projectile Motion

1D Projectile Motion

Having covered some serious ground in statics, it is now time to allow motion into our problems.

Example

Suppose you hold your phone d meters above the ground and drop it. How long will it take the phone to hit the ground and what will its velocity be the instant it hits the ground?

Solution

The only acceleration acting on your phone is straight down towards the earth. You have seen this type of problem both in class and in the course textbook, so let’s take the opportunity to do something weird - we will set our positive axis of motion to point *down towards the Earth*, so that your phone will have acceleration $a = +g$. We will find that you have to be careful when you define "down as up" in this manner.

Using $a = \frac{dv}{dt}$,

$$\frac{dv}{dt} = g \quad (6)$$

Integrating both sides,

$$v = g\,t + C_1 \quad (7)$$

We will define $t = 0$ as the time when you release the phone, at which point $v = 0$. Therefore $C_1 = 0$,

$$v = g\,t \quad (8)$$

Using $v = \frac{dr}{dt}$,

$$\frac{dr}{dt} = g\,t \quad (9)$$

Integrating both sides,

$$r = \frac{1}{2}\,g\,t^2 + C_2 \quad (10)$$

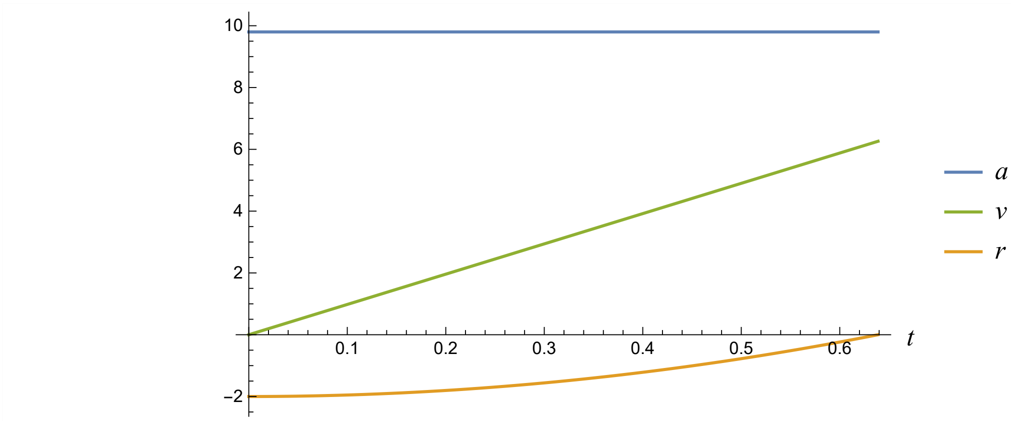
At $t = 0$, the phone is at a height $r = -d$ (remember, $r = 0$ is the boundary of the Earth and we assumed that the direction *towards the Earth* is positive (this weirdness is why people always set the direction away from Earth to be positive!)) which allows us to solve for $C = -d$,

$$r = \frac{1}{2}\,g\,t^2 - d \quad (11)$$

We can now solve for when the object hits the ground ($r = 0$), which yields

$$t = \pm\left(\frac{2\,d}{g}\right)^{1/2} \quad (12)$$

The positive solution is the physical one (can you think of what the negative solution represents?). The graph below plots the position r, velocity v, and the acceleration a for the phone assuming $d = 2\ m$ and $g = 9.8\ \frac{m}{s^2}$.



The time to impact the ground would be $t_{\text{impact}} = 0.64\,s$, and the velocity of your phone would be $v_{\text{impact}} = g\,t_{\text{impact}} = 6.3\,\frac{m}{s}$. So make sure that your phone's case is rated to handle this speed before the next time you try balancing it on your head to check your posture. □

2D Projectile Motion

Your pure-vanilla basic 2D projectile motion problem is extremely similar to the 1D case. Indeed, if you have the projectile shooting straight upwards, then you regain the 1D problem.

Example

A cannon fires a projectile from (0, 0) at an angle θ from the ground with speed v_0. What does the projectile's motion in the air look like?

Solution

We orient the y-axis straight up (away from the Earth) and the x-axis in the direction orthogonal to it in the direction of the projectile's motion. We set $t = 0$ at the moment the cannon fires. We denote the components of velocity by

$$\vec{v} = v_x[t]\,\hat{x} + v_y[t]\,\hat{y} \tag{13}$$

Right after the canon fires, we have the initial conditions

$$v_x[0] = v_0\,\text{Cos}[\theta] \tag{14}$$
$$v_y[0] = v_0\,\text{Sin}[\theta] \tag{15}$$

The only acceleration acting on the projectile is gravitational acceleration $\vec{a} = -g\,\hat{y}$. Thus, by Newton's 1st Law, throughout the projectile's motion $v_x[t]$ remains the same,

$$v_x[t] = v_x[0] = v_0\,\text{Cos}[\theta] \tag{16}$$

In the y-direction, we use $\frac{dv_y}{dt} = a = -g$,

$$\frac{dv_y[t]}{dt} = -g \tag{17}$$

(which is exactly the same as a 1D projectile motion problem). Integrating,

$$v_y[t] = -g\,t + C \tag{18}$$

Applying Equation (15) shows that $C = v_0\,\text{Sin}[\theta]$,

$$v_y[t] = -g\,t + v_0\,\text{Sin}[\theta] \tag{19}$$

Denoting $r_x[t]$ and $r_y[t]$ as the horizontal and vertical distances traveled with $r_x[0] = 0$, $r_y[0] = 0$, $\frac{dr_x[t]}{dt} = v_x[t]$, and $\frac{dr_y[t]}{dt} = v_y[t]$,

$$\frac{dr_x[t]}{dt} = v_x[t] = v_0 \, \mathrm{Cos}[\theta] \tag{20}$$

$$\frac{dr_y[t]}{dt} = v_y[t] = -g\, t + v_0 \, \mathrm{Sin}[\theta] \tag{21}$$

Integrating,

$$r_x[t] = v_0 \, t \, \mathrm{Cos}[\theta] + C_1 \tag{22}$$

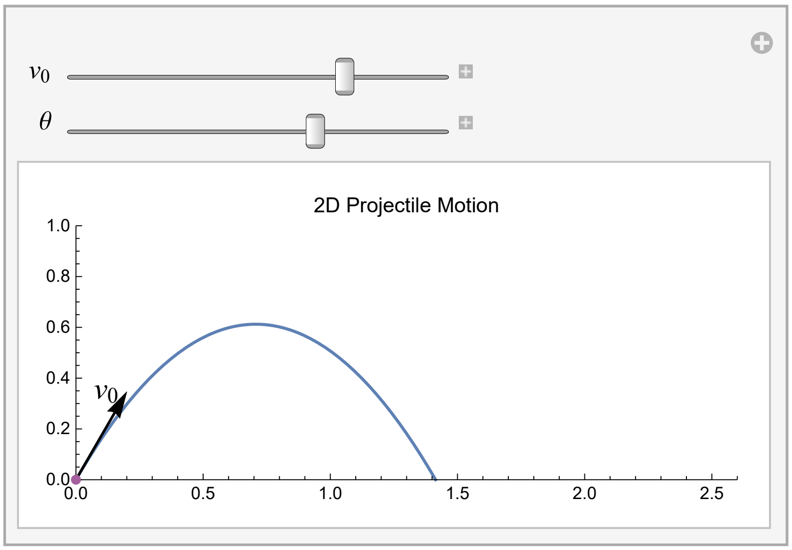
$$r_y[t] = -\frac{1}{2} g \, t^2 + v_0 \, t \, \mathrm{Sin}[\theta] + C_2 \tag{23}$$

Using $r_x[0] = 0$ and $r_y[0] = 0$ allows us to solve for the two constants $C_1 = 0$ and $C_2 = 0$,

$$r_x[t] = v_0 \, t \, \mathrm{Cos}[\theta] \tag{24}$$

$$r_y[t] = -\frac{1}{2} g \, t^2 + v_0 \, t \, \mathrm{Sin}[\theta] \tag{25}$$

We can use *Mathematica* to plot this result by varying t



At $t = 0$, the projectile is launched from the ground and $r_y[t] = 0$. The projectile then flies high into the air because $v_y[t]$ starts off very large, but as time progresses $v_y[t]$ decreases (linearly with time) until it hits 0 at the top of the flight and then becomes negative, bringing the projectile back to Earth.

The projectile's motion is parabolic, and this can be confirmed by solving for t as a function of $r_x[t]$ in Equation (24),

$$t = \frac{r_x[t]}{v_0 \, \mathrm{Cos}[\theta]} \tag{26}$$

and then substituting this result into $r_y[t]$ in Equation (25),

$$r_y[t] = -\frac{1}{2} g \left(\frac{r_x[t]}{v_0 \, \mathrm{Cos}[\theta]}\right)^2 + v \left(\frac{r_x[t]}{v_0 \, \mathrm{Cos}[\theta]}\right) \mathrm{Sin}[\theta] = -\frac{g}{2 \, v_0^2 \, \mathrm{Cos}[\theta]^2} \, r_x^2[t] + \mathrm{Tan}[\theta] \, r_x[t] \tag{27}$$

which is indeed a parabola. We can now easily find the peak height of the projectile's flight by solving Equation (19) for the time at which $v_y[t] = 0$,

$$t_{\text{max height}} = \frac{v_0 \, \mathrm{Sin}[\theta]}{g} \tag{28}$$

and then substituting this time back into Equation (27) for $r_y[t]$ to obtain

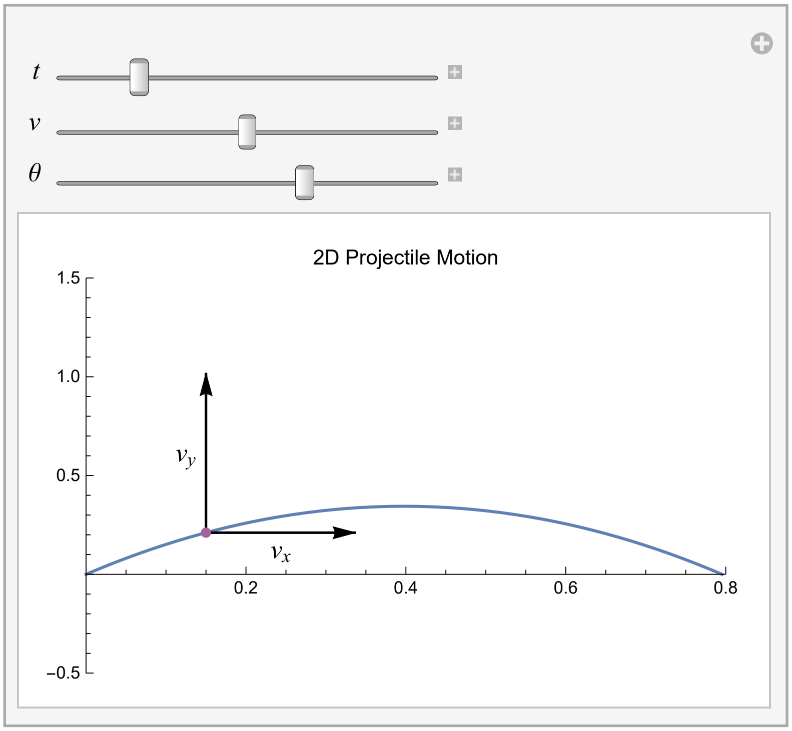
$$r_{y,\text{max height}} = \frac{v_0^2 \, \mathrm{Sin}[\theta]^2}{2\, g} = \frac{v_y[0]^2}{2\, g} \tag{29}$$

We can also find out how far the projectile travels before it splatters all over the ground by solving Equation (27) for its non-zero root when $r_y[t] = 0$,

$$r_{x,\text{max distance}} = \frac{2 \, v_0^2 \, \mathrm{Cos}[\theta] \, \mathrm{Sin}[\theta]}{g} = \frac{2 \, v_x[0] \, v_y[0]}{g} \tag{30}$$

The following graph plots the y-position (r_y) and velocity ($v_y[t]$) of the projectile during its flight. The green arrows show the full velocity vector,

$$v_x[t] = v_0 \,\mathrm{Cos}[\theta] \tag{31}$$
$$v_y[t] = -g\,t + v_0 \,\mathrm{Sin}[\theta] \tag{32}$$



Give yourself a pat on the back. You are now a 2D projectile master! □

Advanced Section: The Cupcake Cannons

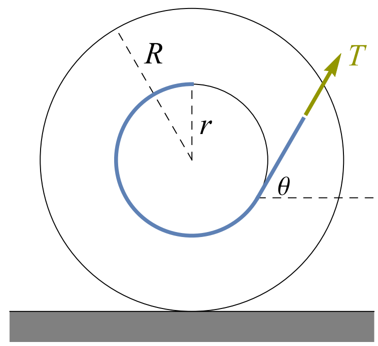
Pulleys

Last time, we covered gravity, friction, and the normal force in several examples. This time, we will focus on the last important force: tension. These and other great problems come from David Morin's book *Introduction to Classical Mechanics with Problems and Solutions*.

A Spool

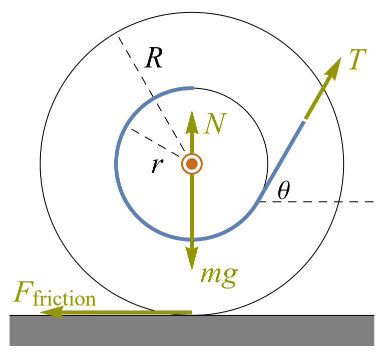
Example

A spool has an inner radius r around which string is wound and an outer radius R on which it can roll. The string is pulled at an angle θ with the horizontal with tension T. Assume that μ is large enough so that the spool does not slip. What value must θ be to prevent the spool from moving?



<u>Solution</u>

All four types of forces apply in this problem: gravity, a normal force, tension, and friction!



Balancing the horizontal and vertical forces,

$$T\,\text{Cos}[\theta] = F_{\text{friction}} \tag{53}$$

$$N + T\,\text{Sin}[\theta] = m\,g \tag{54}$$

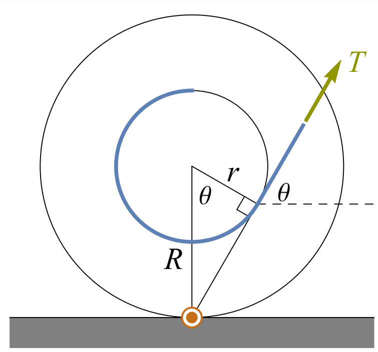
Balancing torques about the center of the spool (so that gravity and the normal force don't contribute),

$$r\,T = R\,F_{\text{friction}} \tag{55}$$

Combining Equations (53) and (55), we find

$$\text{Cos}[\theta] = \frac{r}{R} \tag{56}$$

Often when you find such simple solutions in physics, it indicates that there is a quick and simple method to determine the answer. In this case, a much more clever choice of the torque base point is at the base of the spool, in which case gravity, the normal force, and the frictional force have zero torque. Therefore, the tension force (the only force remaining) must also pass through the base point of the spool in order for it to not contribute any torque.

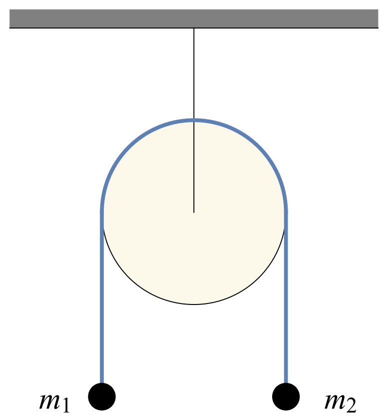


Extending the line of the T force and using simple geometry, we obtain $\text{Cos}[\theta] = \frac{r}{R}$. □

A Pulley

Example

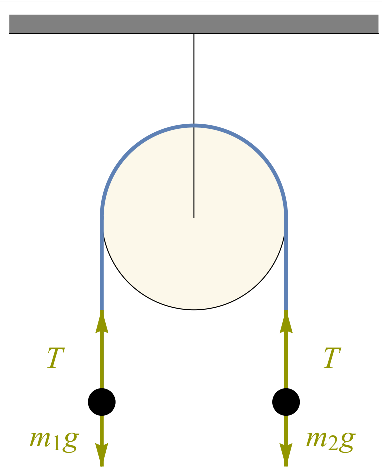
A massless, frictionless pulley has two masses m_1 and m_2 hanging on it by a massless string. Find the acceleration of the masses and the tension in the string.



Solution

The tension T pulls up on both masses. Defining upwards as positive, the accelerations a_1 and a_2 of the masses satisfies

$$T - m_1\, g = m_1\, a_1 \tag{57}$$
$$T - m_2\, g = m_2\, a_2 \tag{58}$$



The rope has fixed length, implying that when mass 1 moves upwards by y_1 then mass 2 moves downwards by $y_2 = -y_1$. Taking two time derivatives,

$$a_2 = -a_1 \tag{59}$$

We can solve this system for T, a_1, and a_2 to obtain

$$T = \frac{2\, m_1\, m_2}{m_1 + m_2}\, g \tag{60}$$
$$a_1 = \frac{m_2 - m_1}{m_2 + m_1}\, g \tag{61}$$

$$a_2 = -\frac{m_2 - m_1}{m_2 + m_1} g \tag{62}$$

Let’s check some special cases:

Case 1: $m_1 = m_2$

$a_1 = a_2 = 0$ and $T = m_1 g$ as expected.

Case 2: $m_1 \ll m_2$

$a_1 = -a_2 = g$ since m_2 is essentially in free fall. The tension $T = 2 m_1 g$ is exactly what is needed to accelerate m_1 upwards with acceleration g.

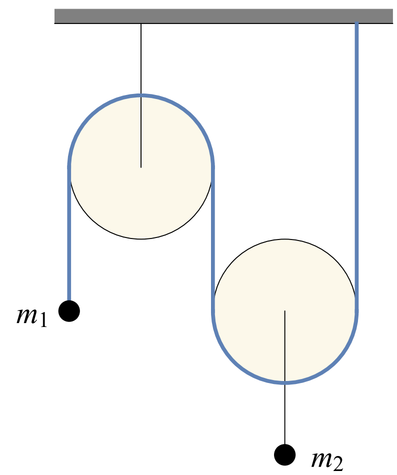
Case 3: $m_1 \gg m_2$

The reverse of Case 2. □

Two Pulleys

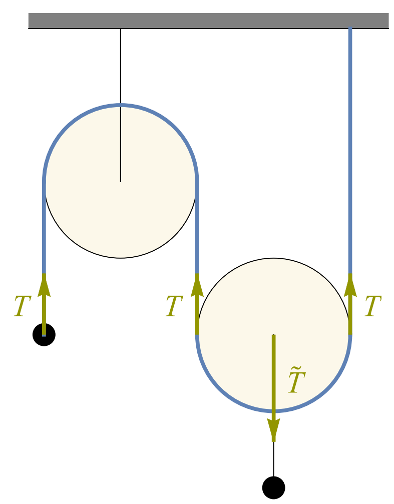
Example

Masses m_1 and m_2 are hung on massless (and therefore frictionless) pulleys and strings. What is the tension in the string and the acceleration of the masses?



Solution

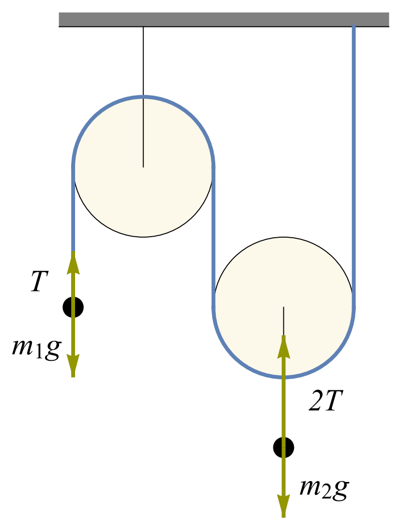
Denote the tension in the rope holding m_1 by T (recall that this is the tension throughout the entire rope because there is no friction). Let’s isolate the forces acting upon the right pulley: the same rope holding up m_1 wraps around this pulley and pulls up on it with tension T from both sides; additionally there is a tension $\tilde{T}$ due to the rope holding up m_2.



Because the pulley is massless, the sum of forces on it must sum to zero (otherwise it would undergo infinite acceleration). Therefore, $\tilde{T} = 2\,T$ is required to balance the forces on this right pulley. Using $\sum \vec{F} = m\,\vec{a}$ on each mass,

$$T - m_1\,g = m_1\,a_1 \tag{63}$$

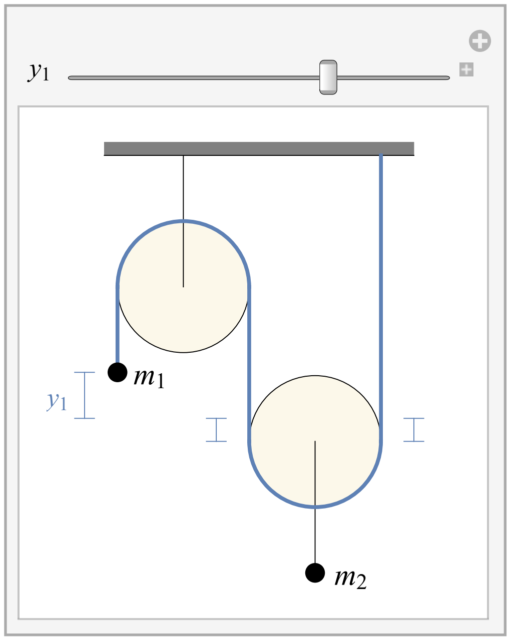
$$2\,T - m_2\,g = m_2\,a_2 \tag{64}$$



where in the second equation of motion we have used the fact that there is tension on either side of the right pulley, and both of these sides pull up on the pulley with tension T. The above two equations are not enough to solve for our three unknowns (a_1, a_2, and T). What is the third equation?

The string has a fixed length! Thus, if m_1 were to rise up by an amount y_1, then m_2 would have to sink by an amount $y_2 = -\frac{1}{2}\,y_1$. Taking two time derivatives of this equation yields

$$a_2 = -\frac{1}{2}\,a_1 \tag{65}$$



We can now solve this system of equations (possibly using *Mathematica*) to obtain

$$T = \frac{3\, m_1\, m_2}{4\, m_1 + m_2}\, g \tag{66}$$

$$a_1 = \frac{2\, m_2 - 4\, m_1}{4\, m_1 + m_2}\, g \tag{67}$$

$$a_2 = \frac{2\, m_1 - m_2}{4\, m_1 + m_2}\, g \tag{68}$$

This result begs for us to check limiting cases (that is when you truly start to understand physics):

Case 1: $m_2 = 2\, m_1$

$a_1 = a_2 = 0$ and $T = m_1\, g$. Static equilibrium is reached if m_2 is twice the mass of m_1 because it has twice as many ropes pulling up on it.

Case 2: $m_1 \ll m_2$

$a_1 = 2\, g$, $a_2 = -g$, and $T = 3\, m_1\, g$. In this case, m_2 is essentially in free fall, and consequently it pulls up m_1 with acceleration $2\, g$ because of the fixed rope length.

Case 3: $m_1 \gg m_2$

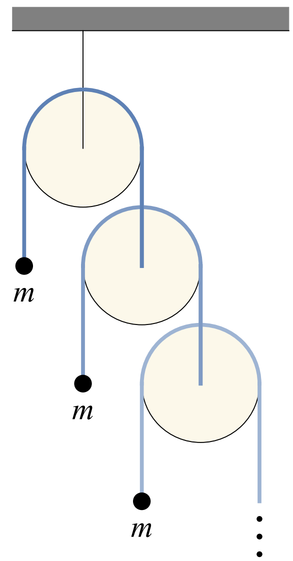
$a_1 = -g$, $a_2 = \frac{1}{2}\, g$, and $T = \frac{3}{4}\, m_2\, g$. In this case, m_1 is essentially in free fall and m_2 just comes along for the ride. □

Advanced Section: Pulley Bonanza!

This problem will blow your mind!

Example

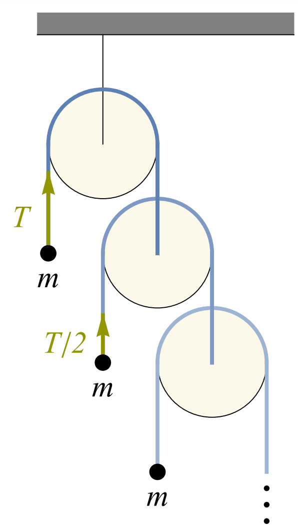
Consider the infinite pulley system below. All the masses m are held fixed and simultaneously released. What is the acceleration of the top mass?



<u>Solution</u>

If the strength of gravity were increased from g to $\lambda\, g$, then the tension must change from T to $\lambda\, T$. Why? Because the only units in this problem are from the mass m (kg) and gravity g ($\frac{m}{s^2}$). The only way to make units of tension (which is the unit of force ($\frac{\text{kg}\cdot m}{s^2}$)) is by multiplying $m\, g$, and hence $T \propto m\, g$. So if we changed $g \to \lambda\, g$, then $T \to \lambda\, T$. Said another way, the ratio $\frac{T}{g}$ will be constant regardless of the particular value of g.

Call the tension in the string holding the top mass T. Because the second pulley is massless, the net force on it must be zero, and therefore the tension in the second rope from the top must be $\frac{T}{2}$.



Define a_1 as *upwards* acceleration of the top mass at time zero. Then a_1 also equals the *downwards* acceleration of the second pulley (by conservation of string length).

Consider the subsystem consisting of all pulleys except for the top one. This system is identical to our original system, except that gravity in this system is effectively $g - a_1$ (this is the feeling of reduced gravity when an elevator starts to descend). Therefore, we know that the $\frac{T}{g}$ in the original system must equal $\frac{T/2}{g-a_1}$ in the subsystem.

Solving $\frac{T}{g} = \frac{T/2}{g-a_1}$ yields $a_1 = \frac{1}{2} g$. Therefore, the top mass accelerates upwards at $\frac{1}{2} g$. □

Another way to solve this problem is to consider a system of N pulleys where a mass m replaces the $N + 1^{\text{th}}$ pulley, and then take the limit as $N \to \infty$. I encourage you to try this rather challenging problem out for yourself!

Advanced Section: Pulleys with Friction

Mathematica Initialization